by going back or recurring each time to the foregoing terms. This is the important pattern of *recursion.*

3.5. Abracadabra

The word "abracadabra" means something like "complicated nonsense." We use the word contemptuously today, but there was a time when it was a magic word, engraved on amulets in mysterious forms (like Fig. 3.2 in some respect), and people believed that such an amulet would protect the wearer from disease and bad luck.

In how many ways can you read the word "abracadabra" in Fig. 3.2? It is understood that we begin with the uppermost A (the north corner) and read down, passing each time to the next letter (southeast or southwest) till we reach the last A (the south corner).

The question is curious. Yet your interest may be really aroused if you notice that there is something familiar behind it. It may remind you of walking or driving in a city. Think of a city that consists of perfectly square blocks, where one-half of the streets run from northwest to southeast and the other streets (or avenues) crossing the former run from northeast to southwest. Reading the magic word of Fig. 3.2 corresponds to a zigzag path in the network of such streets. When you walk along the zigzag path emphasized in Fig. 3.3, you walk ten blocks from the initial A to the final A. There are several other paths which are ten blocks long between these two endpoints in this network of streets, but there is no path that would be shorter. *Find the number of the different shortest paths in the network between the given endpoints*—this is the general, really

```
          A
        B   B
      R   R   R
    A   A   A   A
  C   C   C   C   C
A   A   A   A   A   A
  D   D   D   D   D
    A   A   A   A
      B   B   B
        R   R
          A
```

Fig. 3.2. A magic word.

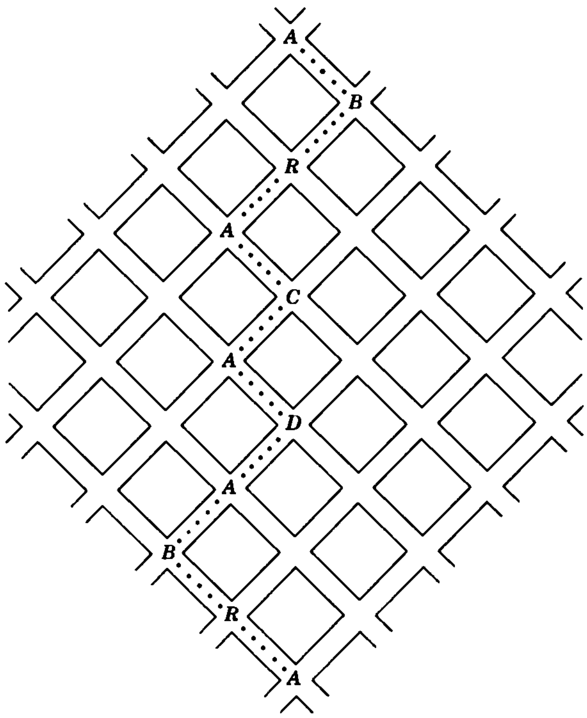

Fig. 3.3. The zigzag path is the shortest.

interesting, problem behind the curious particular problem about the magic word of Fig. 3.2.

A general formulation may have various advantages. It sometimes suggests an approach to the solution, and this happens in our case. *If you cannot solve the proposed problem* about Fig. 3.2 (probably you cannot), *try first to solve some simpler related problem.* At this point the general formulation may help: it suggests trying simpler cases that fall under it. In fact, if the two given corners are close enough to each other in the network (closer than the extreme A's in Fig. 3.3) it is easy to count the different zigzag paths between the two: you can draw each one after the other and survey all of them. Listen to this suggestion and pursue it systematically. Start from the point A and go downward. Consider first the points that you can reach by walking one block, then those to which you have to walk two blocks, then those which are three or four or more blocks away. Survey and count for each point the shortest zigzag paths

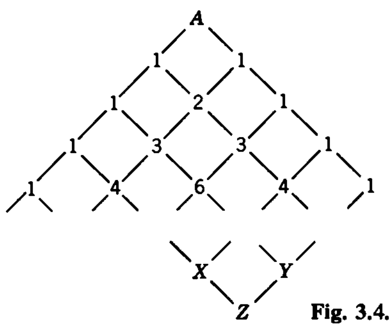

Fig. 3.4. Count the number of shortest zigzag paths.

that connect it with A. In Fig. 3.4 a few numbers so obtained are marked (but you should have obtained these numbers and a few more by yourself—check them at least). Observe these numbers—do you notice something?

If you have enough previous knowledge you may notice many things. Yet even if you have never before seen this array of numbers displayed by Fig. 3.4 you may notice an interesting relation: any number in Fig. 3.4 that is different from 1 is the sum of two other numbers in the array, of its northwest and northeast neighbors. For instance,

$$4 = 1 + 3, \qquad 6 = 3 + 3$$

You may discover this law by observation as a naturalist discovers the laws of his science by observation. Yet, after having discovered it, you should ask yourself: Why is that so? What is the reason?

The reason is simple enough. Consider three corners in your network, the points X, Y, and Z, the relative position of which is shown by Fig. 3.4: X is the northwest neighbor and Y the northeast neighbor of Z. If we wish to reach Z coming from A along a shortest path in the network, we must pass either through X or through Y. Once we have reached X, we can proceed hence to Z in just one way, and the same is true for proceeding from Y to Z. Therefore, the *total number of shortest paths from A to Z* is a sum of two terms: it *equals the number of shortest paths from A to X added to the number of those from A to Y.* This explains fully our observation and proves the general law.

Having clarified this basic point, we can extend the array of numbers in Fig. 3.4 by simple additions till we obtain the larger array in Fig. 3.5, the south corner of which yields the desired answer: we can read the magic word in Fig. 3.2 in exactly 252 different ways.

3.6. The Pascal triangle

By now the reader has probably recognized the numbers and their peculiar configuration which we have examined in the foregoing section. The numbers in Fig. 3.4 are *binomial coefficients* and their triangular arrangement is usually called the *Pascal triangle*. (Pascal himself called it the "arithmetical triangle.") Further lines can be added to the triangle of Fig. 3.4 and, in fact, it can be extended indefinitely. The array in Fig. 3.5 is a square piece cut out of a larger triangle.

Some of the binomial coefficients and their triangular arrangement can be found in the writings of other authors before Pascal's *Traité du triangle arithmétique*. Still, the merits of Pascal in this matter are quite sufficient to justify the use of his name.

(1) We have to introduce a suitable *notation* for the numbers contained in the Pascal triangle; this is a step of major importance. For us each number attached to a point of this triangle has a geometric meaning: it indicates the number of different shortest zigzag paths from the apex of the triangle to that point. Each of these paths passes along the same number of blocks, let us say n blocks. Moreover, all these paths agree in the number of blocks described in the southwesterly direction and in the number of those in the southeasterly direction. Let l and r stand for these numbers, respectively (l to the left and r to the right—of course, downward in both cases). Obviously

$$n = l + r$$

If we give any two of the three numbers n, l, and r, the third is fully determined and so is the point to which they refer. (In fact, l and r are the rectangular coordinates of the point with respect to a system the origin of which is the apex of the Pascal triangle; one of the axes points southwest,

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
6 15 20 15 6
21 35 35 21
56 70 56
126 126
252

Fig. 3.5. A square from a triangle.

the other southeast.) For instance, for the last A of the path shown in Fig. 3.3

$$l = 5, \qquad r = 5, \qquad n = 10$$

and for the second B of the same path

$$l = 5, \qquad r = 3, \qquad n = 8$$

We shall denote by $\binom{n}{r}$ (this notation is due to Euler) the number of shortest zigzag paths from the apex of the Pascal triangle to the point specified by n (total number of blocks) and r (blocks to the right downward). For instance, see Fig. 3.5,

$$\binom{8}{3} = 56, \qquad \binom{10}{5} = 252$$

The symbols for the numbers contained in Fig. 3.4 are assembled in Fig. 3.6. The symbols with the same number upstairs (the same n) are horizontally aligned (along the nth "base"—the base of a right triangle). The symbols with the same number downstairs (the same r) are obliquely aligned (along the rth "avenue"). The fifth avenue forms one of the sides of the square in Fig. 3.5—the opposite side is formed by the 0th avenue (but you may call it the borderline, or Riverside Drive, if you prefer to do so). The fourth base is emphasized in Fig. 3.4.

(2) Besides the geometric aspect, the Pascal triangle also has a computational aspect. All the numbers along the boundary (0th street, 0th avenue, and their common starting point) are equal to 1 (it is obvious that

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}$$

$$\begin{array}{ccc}
\binom{n}{r-1} & & \binom{n}{r} \\
 & \binom{n+1}{r} &
\end{array}$$

Fig. 3.6. Symbolic Pascal triangle.

there is just one shortest path to these street corners from the starting point). Therefore,

$$\binom{n}{0} = \binom{n}{n} = 1$$

It is appropriate to call this relation the *boundary condition* of the Pascal triangle.

Any number inside the Pascal triangle is situated along a certain horizontal row, or base. We compute a number of the $(n + 1)$th base by going back, or recurring, to two neighboring numbers of the nth base:

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$$

see Fig. 3.6. It is appropriate to call this equation the *recursion formula* of the Pascal triangle.

From the computer's standpoint the numbers $\binom{n}{r}$ are determined (or defined, if you wish) by the recursion formula and the boundary condition of the Pascal triangle.

3.7. Mathematical induction

When we compute a number in the Pascal triangle by using the recursion formula, we have to rely on the previous knowledge of two numbers of the foregoing base. It would be desirable to have a scheme of computation independent of such previous knowledge. There is a well-known formula, which we shall call the *explicit formula* for binomial coefficients, that yields such an independent computation:

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{1 \cdot 2 \cdot 3 \cdots r}$$

Pascal's treatise contains the explicit formula (stated in words, not in our modern notation). Pascal does not say how he has discovered it and we shall not speculate too much how he might have discovered it. (Perhaps he just guessed it first—we often find such things by observation and tentative generalization of the observed; see the remark in the solution of ex. 3.39.) Yet Pascal gives a remarkable proof for the explicit formula and we wish to devote our full attention to his method of proof.[4]

We need a preliminary remark. The explicit formula does not apply,

[4] Cf. Pascal's *Œuvres l.c.* footnote 3, pp. 455–464, especially pp. 456–457. The following presentation takes advantage of modern notation and modifies less essential details.